Lecture 9

Efficient Observation of Random Phenomena

Tokyo Polytechnic University
The 21st Century Center of Excellence Program

Yukio Tamura

POD

- Proper Orthogonal Decomposition
- Stochastic Representation of Factor Analysis
- Karhunen-Loève Decomposition

POD

- Find a deterministic coordinate function $\phi(x,y)$ that best correlates with all the elements of a set of randomly fluctuating wind pressure fields $p\,(x,y,t)$.
- $\phi(x,y)$ is derived to maximize the projection from the wind pressure field $p\,(x,y,t)$ to the deterministic coordinate function $\phi(x,y)$.

Maximization of Projection

Realize from the probabilistic standpoint:

$$\iint p(x,y,t)\,\phi(x,y)\,dxdy \quad = \max$$

By normalizing:

$$\frac{\iint p(x,y,t)\,\phi(x,y)\,dxdy}{\{\iint \phi(x,y)^2\,dxdy\}^{1/2}} = \max$$

Maximization of Projection

$p(x,y,t)$ can take positive or negative values

→ Maximization is made by a mean square method

$$\frac{\overline{\iint p(x,y,t)\phi(x,y)\,dxdy \iint p(x',y',t)\phi(x',y')\,dx'dy'}}{\iint \phi(x,y)^2\,dxdy} = \max$$

→ Eigenvalue Problem

$$\iint R_p(x,y,x',y')\,\phi(x',y')\,dx'dy' = \lambda\,\phi(x,y)$$

$R_p(x,y,x',y')$: Spatial Correlation of $p(x,y,t)$

Uniformly Distributed $dx \times dy$

- Matrix Form

$$[R_p]\{\phi\} = \lambda\{\phi\}$$

$[R_p]$: Spatial Correlation Matrix ($M \times M$ Square Matrix)

$\{\phi\}$: Eigenvector of Spatial Correlation Matrix $[R_p]$

λ : Eigenvalue of Spatial Correlation Matrix $[R_p]$

Fluctuating Pressure Field $p(x,y,t)$

$$p(x,y,t) = \sum_{m=1}^{M} a_m(t)\,\phi_m(x,y)$$

$$a_m(t) = \frac{\iint p(x,y,t)\,\phi_m(x,y)\,dxdy}{\iint \phi_m(x,y)^2\,dxdy}$$

: m-th Principal Coordinate

$\phi_m(x,y)$: m-th Eigenvector (Eigen Mode)

Correlation Between Principal Coordinates

No correlation between different modes :

$$\overline{a_m(t)\,a_n(t)} = 0 \quad (m \neq n)$$

Mean square of the m-th principal coordinate :

$$\overline{a_m(t)^2} = \lambda_m = \text{Eigenvalue}$$

Mean Square of Wind Pressure

Mean-square of wind pressure at point(x,y)

$$\overline{p(x,y,t)^2} = \sum_{m=1}^{M} \lambda_m \phi_m(x,y)^2$$

Field-total sum of mean-square wind pressures

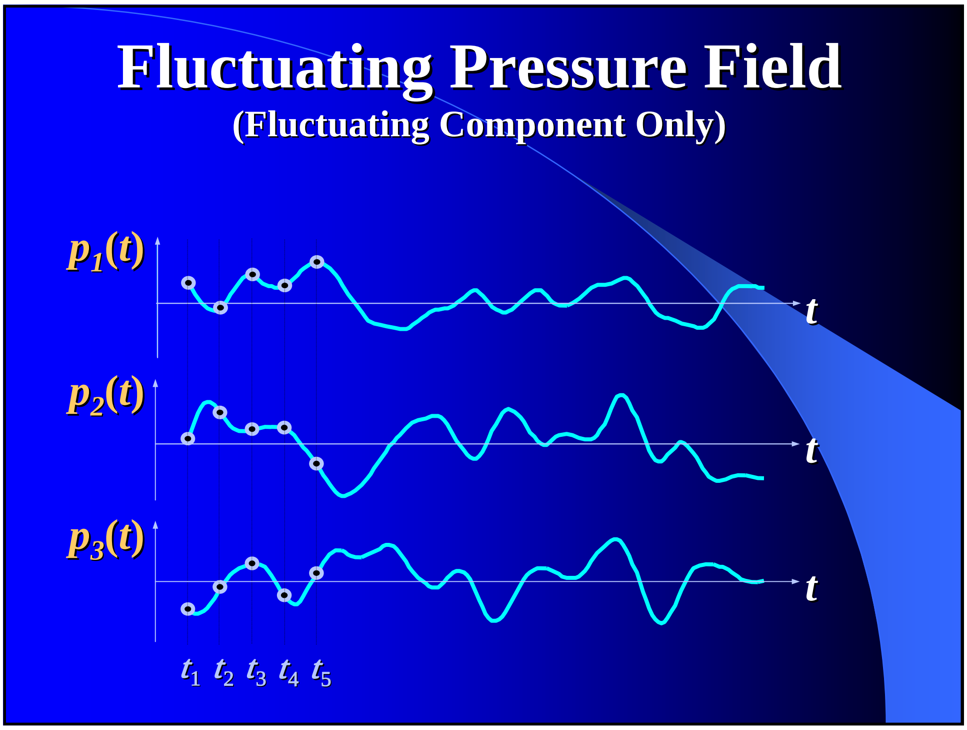
$$\iint \overline{p(x,y,t)^2}\, dxdy = \sum_{m=1}^{M} \lambda_m$$

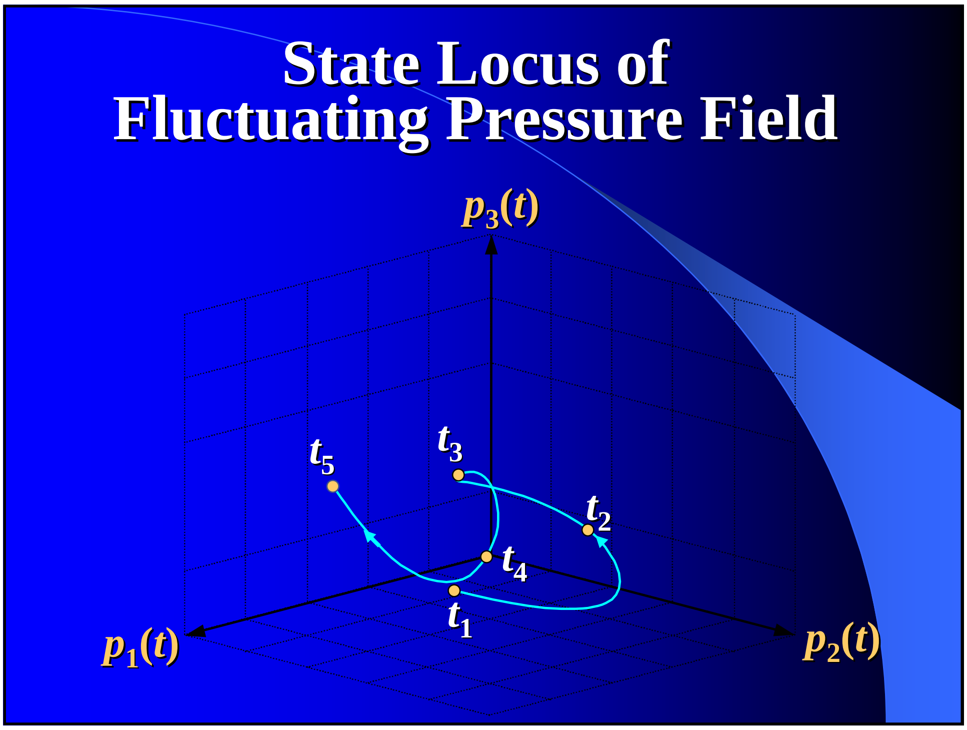
$$= \sum_{m=1}^{M} \overline{a_m(t)^2}$$

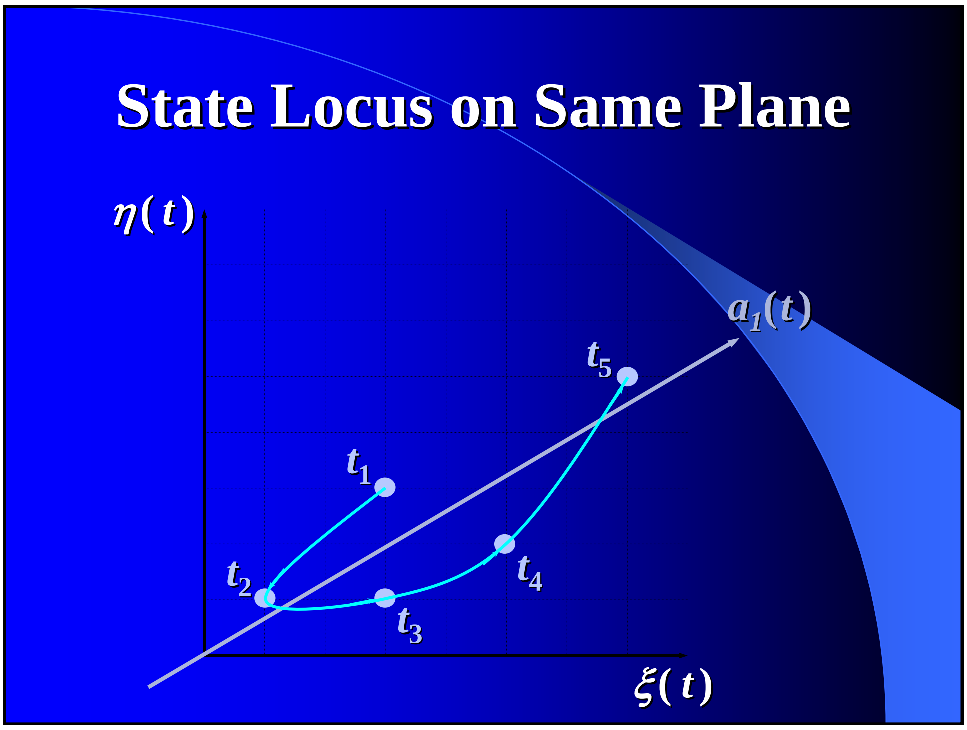
Reconstruction in Lower Modes

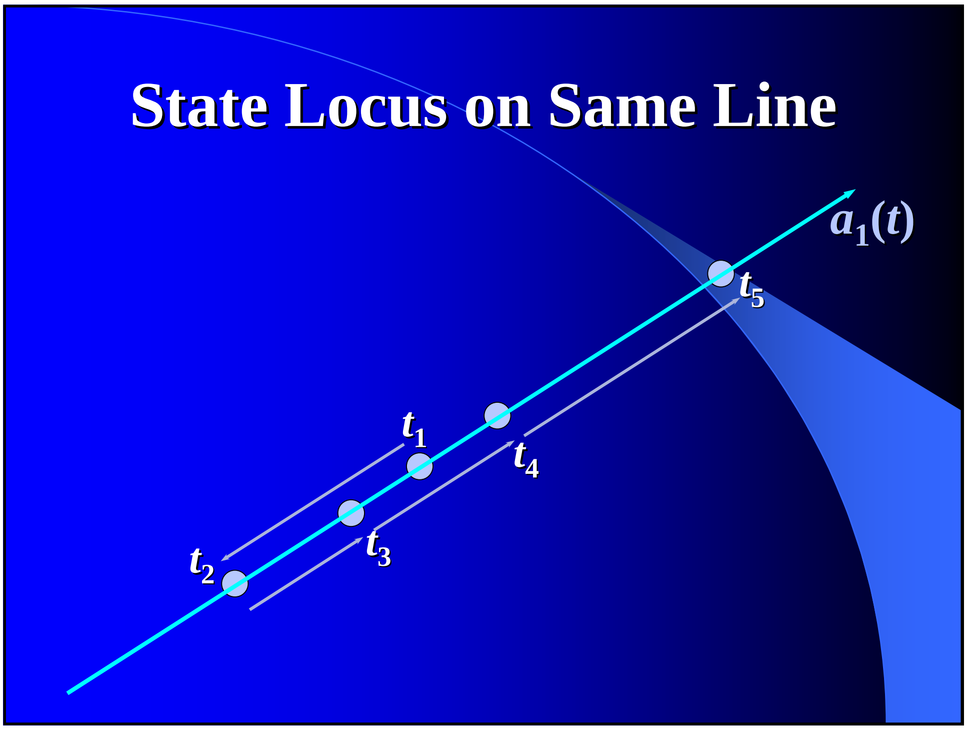
$$\hat{p}(x,y,t) \approx \sum_{m=1}^{N} a_m(t)\,\phi_m(x,y)$$

$$(N \le M)$$

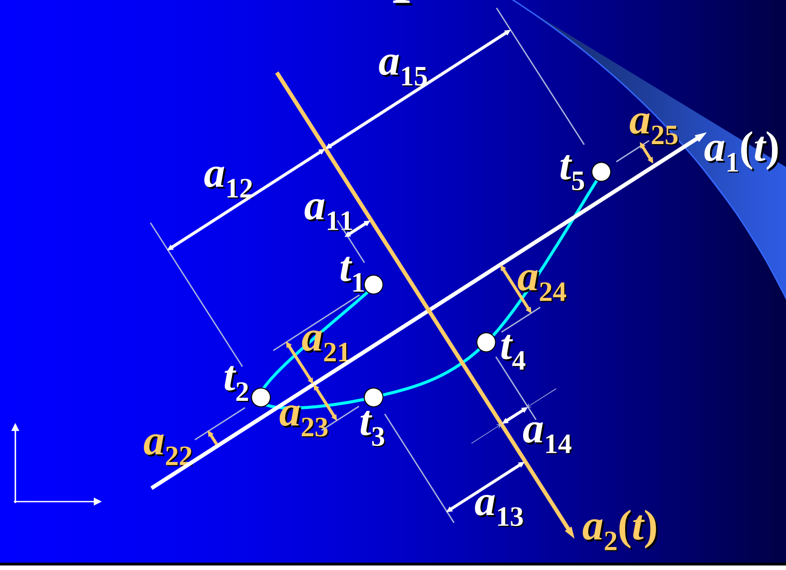

Fluctuating Pressure Field
(Fluctuating Component Only)
p1(t)
p2(t)
p3(t)
t
t
t
t1 t2 t3 t4 t5


State Locus of
Fluctuating Pressure Field
p3(t)
p1(t)
p2(t)
t1
t2
t3
t4
t5


State Locus on Same Plane
η(t)
a1(t)
t5
t1
t2
t3
t4
ξ(t)


State Locus on Same Line
a1(t)
t5
t1
t4
t3
t2

Projection of State Locus onto Principal Coordinate



Maximization of Projection

- Maximization of variance

$$\sigma_{a1}^2 = \frac{1}{M}\sum_{j=1}^{M} a_{1j}^2$$

- not maximization of mean square

  → analyze "fluctuating component" only (zero mean)

Principal Coordinates $a_1(t), a_2(t), a_3(t)$

- **Linear combination of original (physical) coordinates** $p_1(t), p_2(t), p_3(t)$

$$a_1(t) = \phi_{11} p_1(t) + \phi_{12} p_2(t) + \phi_{13} p_3(t)$$
$$a_2(t) = \phi_{21} p_1(t) + \phi_{22} p_2(t) + \phi_{23} p_3(t)$$
$$a_3(t) = \phi_{31} p_1(t) + \phi_{32} p_2(t) + \phi_{33} p_3(t)$$

$$\{a\} = [\phi]\{p\}$$

Coordinate Transformation Matrix $[\phi]$

$$[\phi] = [\phi_{ij}]$$

***m*-th row vector**

$$\{\phi_m\} = \{\phi_{m1}, \phi_{m2}, \phi_{m3}\}^T$$

→ *m*-th Eigenvector

Maximization of Variance of the 1st Mode Principal Coordinate $a_1(t)$

- Assumption

- **No correlation** between principal coordinates $a_1(t)$, $a_2(t)$, $a_3(t)$
- **Unit norm** of eigenvector

$$\phi_{m1}^2 + \phi_{m2}^2 + \phi_{m3}^2 = 1$$

Variance of 1st Principal Coordinate $a_1(t)$

$$\begin{aligned}
\sigma_{a1}^2 &= \overline{\{a_1(t)\}^2} \\
&= \overline{\{\phi_{11} p_1(t) + \phi_{12} p_2(t) + \phi_{13} p_3(t)\}^2} \\
&= \phi_{11}^2 \sigma_1^2 + \phi_{12}^2 \sigma_2^2 + \phi_{13}^2 \sigma_3^2 \\
&\quad + 2\phi_{11}\phi_{12}\sigma_{12} + 2\phi_{11}\phi_{13}\sigma_{13} \\
&\quad + 2\phi_{12}\phi_{13}\sigma_{23}
\end{aligned}$$

σ_{a1}^2 : Variance of the 1st principal coordinate $a_1(t)$

σ_m^2 : Variance of fluctuating pressure $p_m(t)$

σ_{mn} : Covariance of $p_m(t)$ and $p_n(t)$

Lagrange's Method of Indeterminate Coefficients

Maximizing variance σ_{a1}^2 of $a_1(t)$ is equivalent to maximizing the following value L:

$$L = \sigma_{a1}^2 + \lambda\,(\phi_{11}^2 + \phi_{12}^2 + \phi_{13}^2 - 1)$$

λ : constant

$$(\because\ \phi_{11}^2 + \phi_{12}^2 + \phi_{13}^2 = 1)$$

Differential Coefficient of L by ϕ_{1m}

$$\boxed{\frac{\partial L}{\partial \phi_{1m}} = 0}$$

$\frac{\partial L}{\partial \phi_{11}}$

$$(\sigma_1^2 - \lambda)\,\phi_{11} + \sigma_{12}\,\phi_{12} + \sigma_{13}\,\phi_{13} = 0 \quad \text{(a)}$$

$\frac{\partial L}{\partial \phi_{12}}$

$$\sigma_{21}\,\phi_{11} + (\sigma_2^2 - \lambda)\,\phi_{12} + \sigma_{23}\,\phi_{13} = 0 \quad \text{(b)}$$

$\frac{\partial L}{\partial \phi_{13}}$

$$\sigma_{31}\,\phi_{11} + \sigma_{32}\,\phi_{12} + (\sigma_3^2 - \lambda)\,\phi_{13} = 0 \quad \text{(c)}$$

Condition for Non-trivial Solution

Determinant of Coefficients = 0

$$\begin{vmatrix} (\sigma_1^2-\lambda) & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & (\sigma_2^2-\lambda) & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & (\sigma_3^2-\lambda) \end{vmatrix} = 0 \qquad \text{(d)}$$

→ Eigenvalue Problem of Matrix $[R_p]$

Covariance Matrix

$$[R_p] = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 \end{bmatrix}$$

σ_i^2 : Variance

$\sigma_{ij} = \sigma_{ji}$: Covariance

Equation (d) has three roots:

$$\lambda = \lambda_1,\ \lambda_2,\ \lambda_3$$

$$(\lambda_1 \geq \lambda_2 \geq \lambda_3)$$

Eq.(a)×ϕ_{11} +Eq.(b)×ϕ_{12}+Eq.(c)×ϕ_{13}

$$\underbrace{\boxed{\begin{array}{l}\phi_{11}{}^2\sigma_1{}^2 + \phi_{12}{}^2\sigma_2{}^2 + \phi_{13}{}^2\sigma_3{}^2 \\ +2\phi_{11}\phi_{12}\sigma_{12} + 2\phi_{11}\phi_{13}\sigma_{13} + 2\phi_{12}\phi_{13}\sigma_{23}\end{array}}}_{\sigma_{a1}{}^2}$$

$$-\lambda\,(\underbrace{\phi_{11}{}^2 + \phi_{12}{}^2 + \phi_{13}{}^2}_{=1}) = 0$$

$$\rightarrow \quad \sigma_{a1}{}^2 = \lambda \quad \rightarrow \text{Max. } \lambda_1$$

Variance of Principal Coordinate = Eigenvalue

Orthogonality of Eigenvectors of Matrix

$$\{\phi_m\}^T\{\phi_n\} = \Sigma\ \phi_{mj}\phi_{nj}$$
$$= \phi_{m1}\phi_{n1} + \phi_{m2}\phi_{n2} + \phi_{m3}\phi_{n3}$$
$$= \delta_{mn}$$

δ_{mn} **: Kronecker's Delta**

$$= \begin{cases} 1 & , m = n \\ 0 & , m \neq n \end{cases}$$

Reconstruction of Pressure Field

- **Reconstruction by principal coordinates $a_1(t)$, $a_2(t)$, $a_3(t)$ and eigenvectors**

$$p_1(t) = \phi_{11}\, a_1(t) + \phi_{21}\, a_2(t) + \phi_{31}\, a_3(t)$$
$$p_2(t) = \phi_{12}\, a_1(t) + \phi_{22}\, a_2(t) + \phi_{32}\, a_3(t)$$
$$p_3(t) = \phi_{13}\, a_1(t) + \phi_{23}\, a_2(t) + \phi_{33}\, a_3(t)$$

$$\{p\} = [\phi]^T\{a\}$$

Proportion of *m*-th Mode

Proportion of *m*-th Principal Coordinate

$$c_m = \frac{\text{Variance of } m\text{-th Principal Coordinate}}{\text{Variance of Original Pressure Field}}$$

$$= \frac{\sigma_{am}^{\ 2}}{\sigma_1^{\ 2} + \sigma_2^{\ 2} + \sigma_3^{\ 2}}$$

$$= \frac{\sigma_{am}^{\ 2}}{\sigma_{a1}^{\ 2} + \sigma_{a2}^{\ 2} + \sigma_{a3}^{\ 2}}$$

$$= \frac{\lambda_m}{\lambda_1 + \lambda_2 + \lambda_3}$$

Proportion and Cumulative Proportion

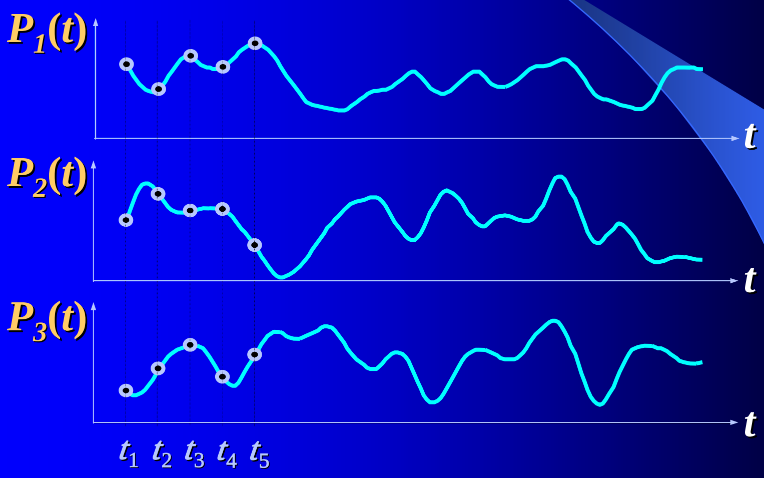
Proportion of *m*-th Principal Coordinate

$$c_m = \frac{\lambda_m}{\sum_{m=1}^{M} \lambda_m}$$

Cumulative Proportion up to *N*-th Mode

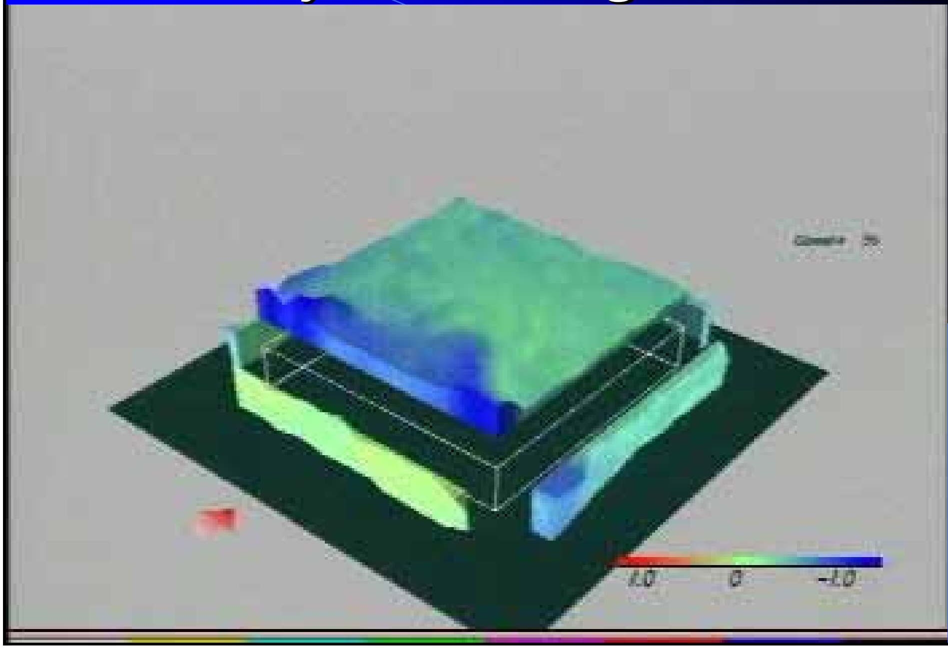
$$C_N = \sum_{m=1}^{N} c_m$$

Fluctuating Pressure Field



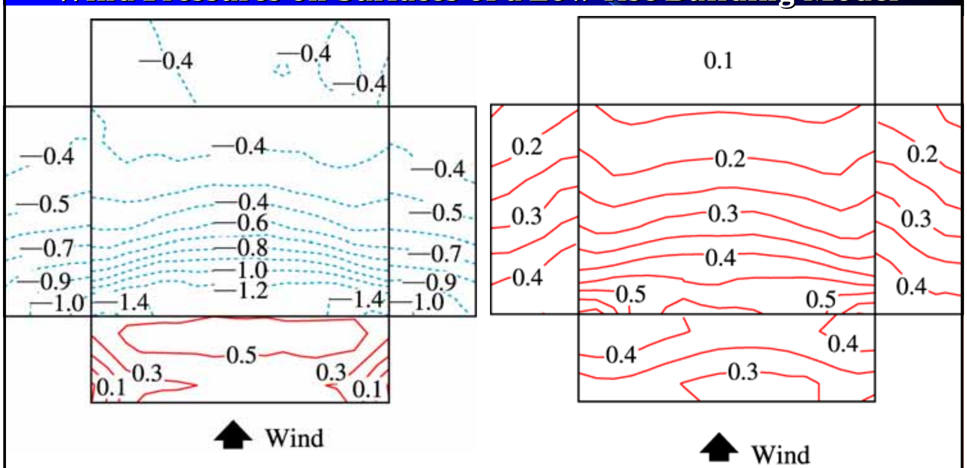
Low-rise Building Model

Randomly Fluctuating Pressures



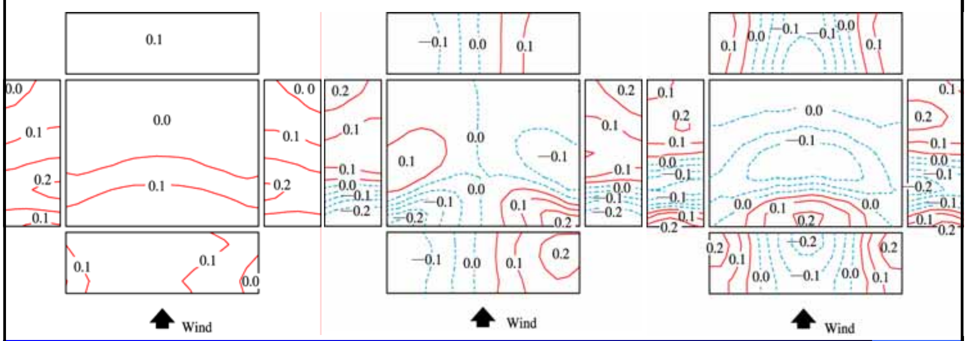
Mean and Standard Deviation of Fluctuating Pressures

Wind Pressures on Surfaces of a Low-rise Building Model



Mean Pressure C_{pmean} **Standard Deviation C_p'**

Eigenvectors
of the Lowest Three Modes

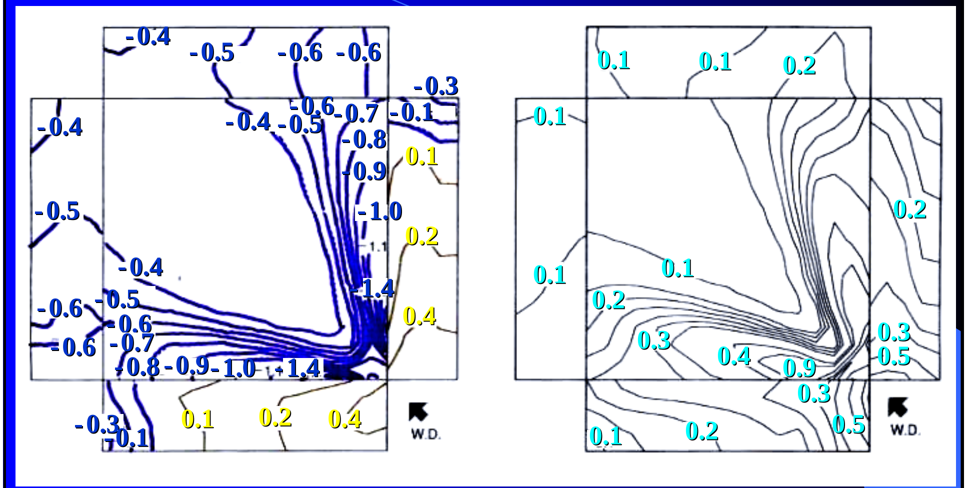


1st Mode 2nd Mode 3rd Mode

Eigenvalues, Proportions and Cumulative Proportions

Wind Pressures on Surfaces of a Low-rise Building Model

Mode	Eigenvalue	Proportion (%)	Cumulative Proportion (%)
1st	1411	40.20	40.20
2nd	295	8.40	48.60
3rd	224	6.39	54.99
4th	175	4.98	59.97
5th	128	3.66	63.63
6th	102	2.91	66.54
7th	80	2.29	68.83
8th	75	2.12	70.95
9th	61	1.74	72.69
10th	53	1.51	74.20

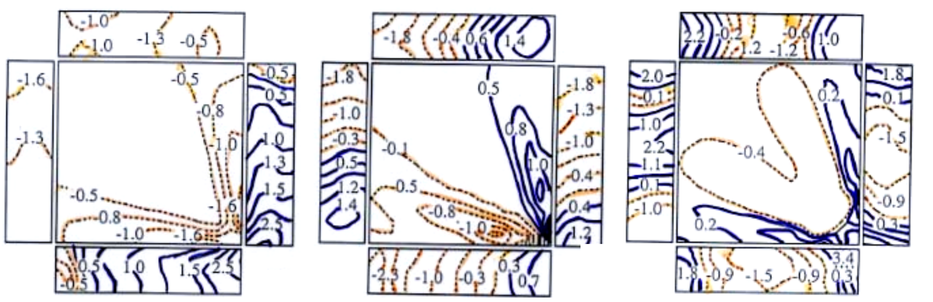


(a) Mean pressure coefficient C_p

(b) Fluctuating pressure coefficient C_p'

Pressure distributions on a low-rise building model (θ = 45°, $D : B : H$ = 4 : 4 : 1)

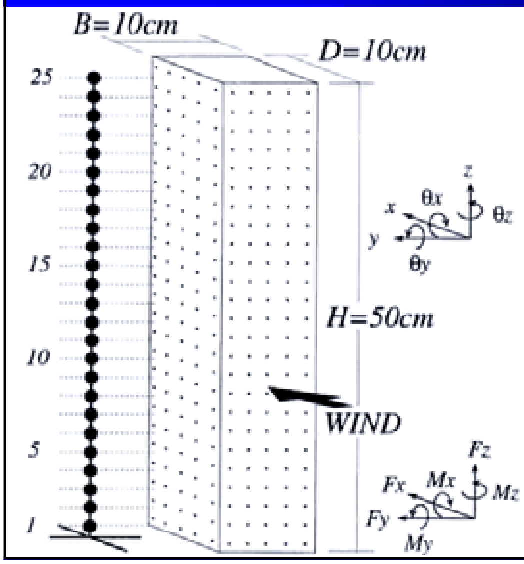
Lowest Three Eigenvectors



1st Mode 2nd Mode 3rd Mode

High-rise Building Model

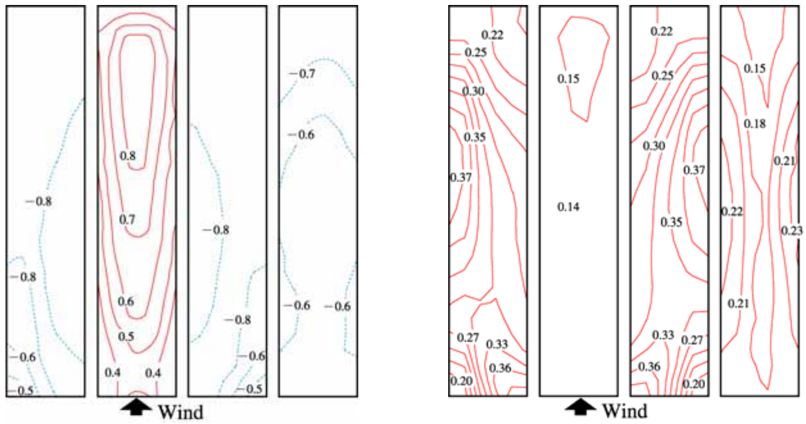
Pressure Measurement Model and Analytical Lumped Mass Model



- Length Scale = 1/400
- Mean Wind Speed Profile $\alpha = 1/6$
- 500 Pressure Taps
- Δt = 0.00128 sec
- T = 42 sec (32,768 samples)

Mean and Standard Deviation of Fluctuating Pressures

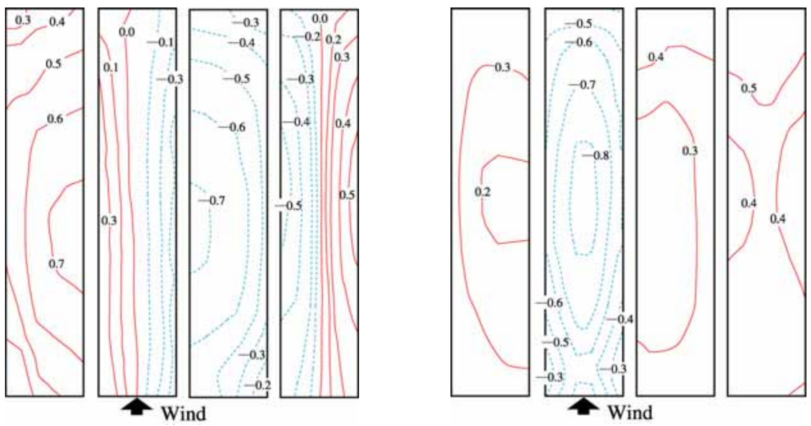
Wind Pressures on Surfaces of a High-rise Building Model



Mean Pressure | Standard Deviation

Eigenvectors of Lowest Two Modes

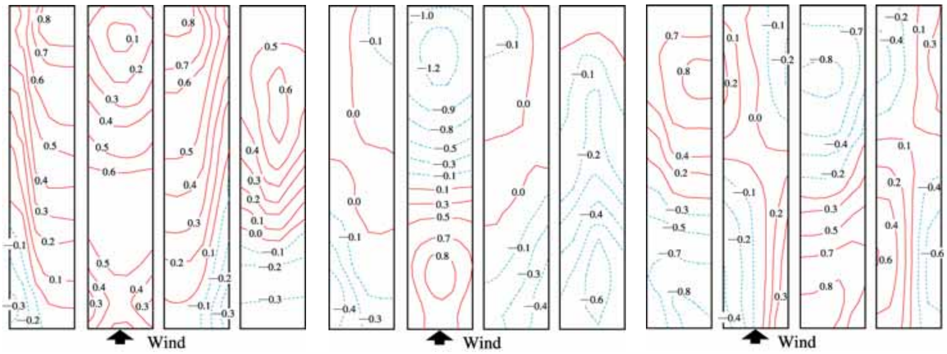
Fluctuating pressures acting on a high-rise building model



1st Mode | 2nd Mode

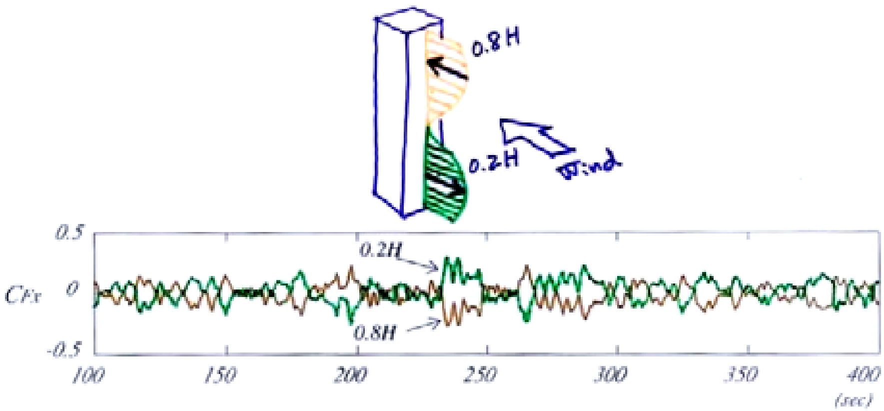
Eigen Vectors of 3rd, 4th and 5th Modes

Fluctuating pressures acting on a high-rise building model



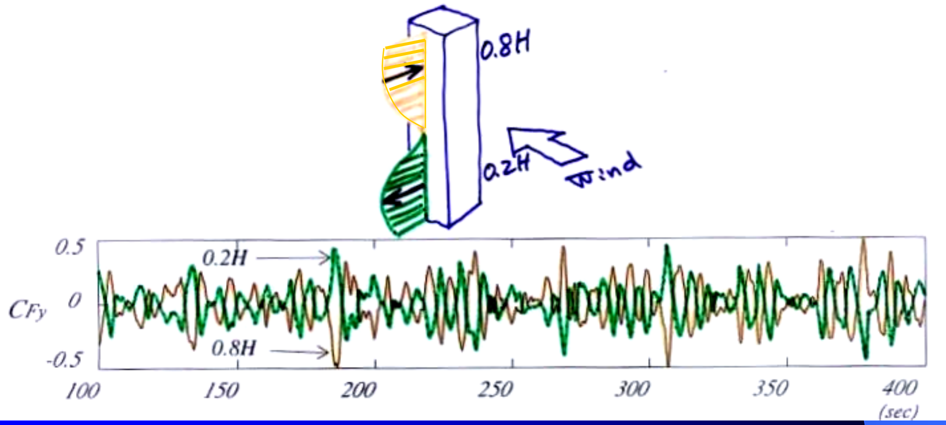
3rd Mode **4th Mode** **5th Mode**

4th Mode



Along-wind force coefficients at 0.2*H* and 0.8*H* by 4th mode (Suburban flow, α=1/6)

5th Mode



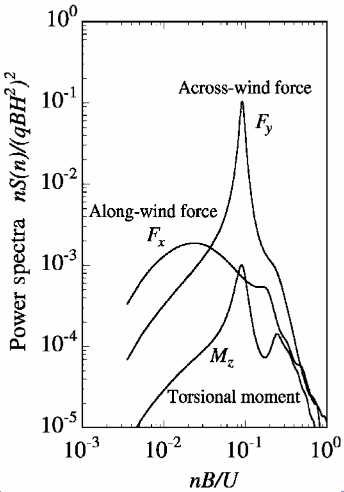
Along-wind force coefficients at 0.2*H* and 0.8*H* by 5th mode (Suburban flow, α=1/6)

Eigenvalues, Proportions and Cumulative Proportions

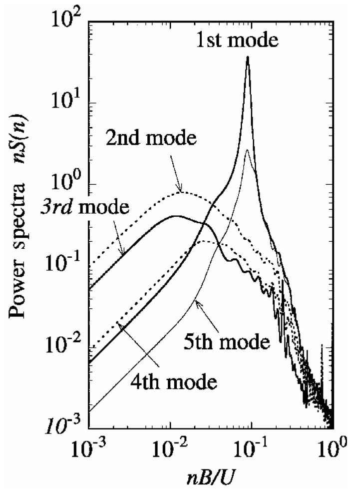
Wind Pressures on Surfaces of a High-rise Building Model

Mode	Eigenvalue	Proportion (%)	Cumulative Proportion (%)
1st	132.00	26.30	26.30
2nd	83.70	16.70	43.00
3rd	32.60	6.51	49.51
4th	25.80	5.16	54.67
5th	25.20	5.04	59.71
-	-	-	-
10th	7.19	1.44	71.43
-			
50th	0.74	0.15	89.24
-	-	-	-
100th	0.26	0.05	93.47
-	-	-	-
300th	0.06	0.01	98.50
-	-	-	-
500th	0.01	0.00	100.00

Power Spectral Densities of Wind Forces

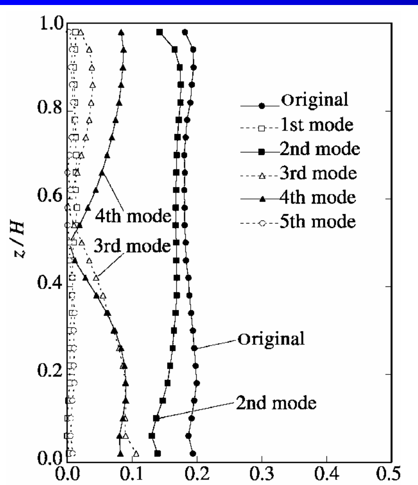


Generalized Wind Forces



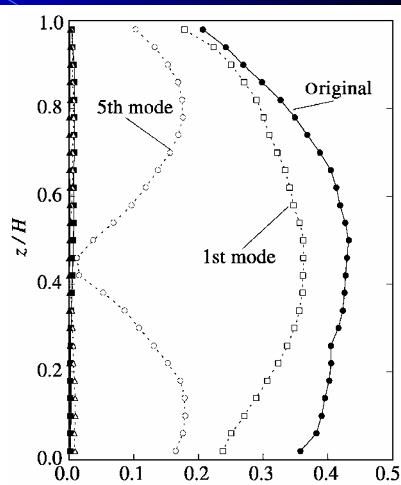
Lowest Five Principal Coordinates

Fluctuating Wind Force Coefficients by Each Mode ($\alpha = 1/6$)



rms C_{Fx}

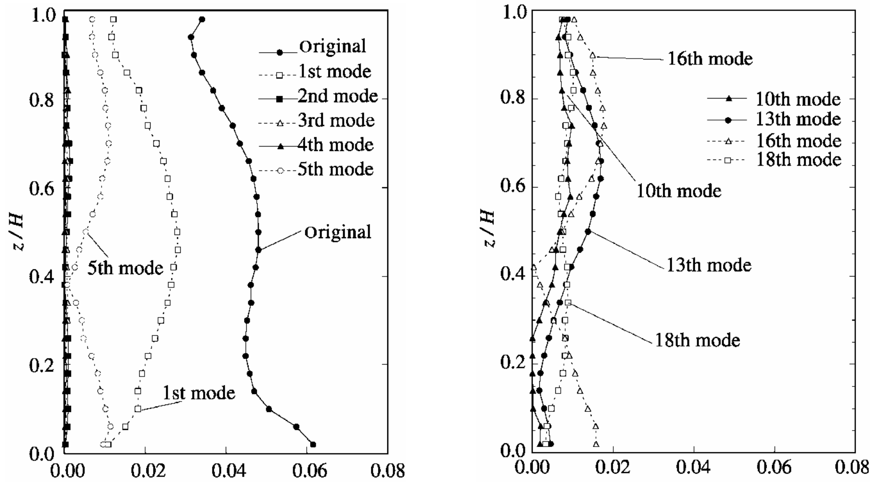
Along-wind Force



rms C_{Fy}

Across-wind Force

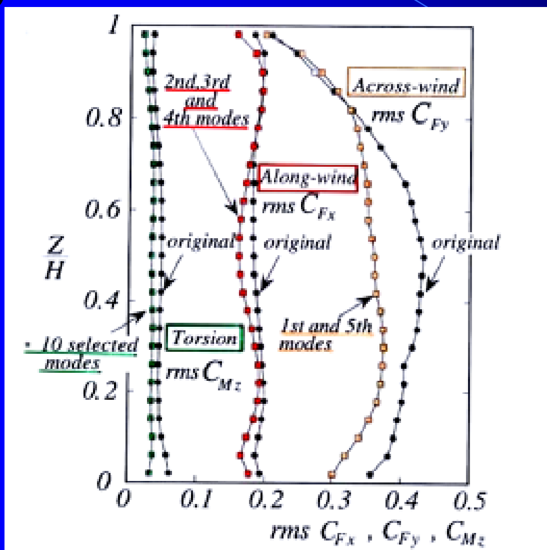
Fluctuating Wind Force Coefficients by Each Mode (α = 1/6)



rms C_{Mz} rms C_{Mz}

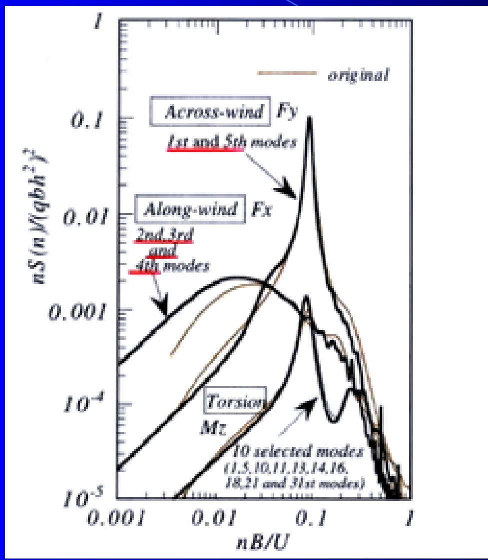
Torsional Moment

RMS Force Coefficients Reconstructed by Selected Dominant Modes (α = 1/6)



1,5,10,11,13,14,16,18, 21 and 31st Modes

Power Spectra of Generalized Wind Forces Reconstructed by Selected Dominant Modes (α = 1/6)

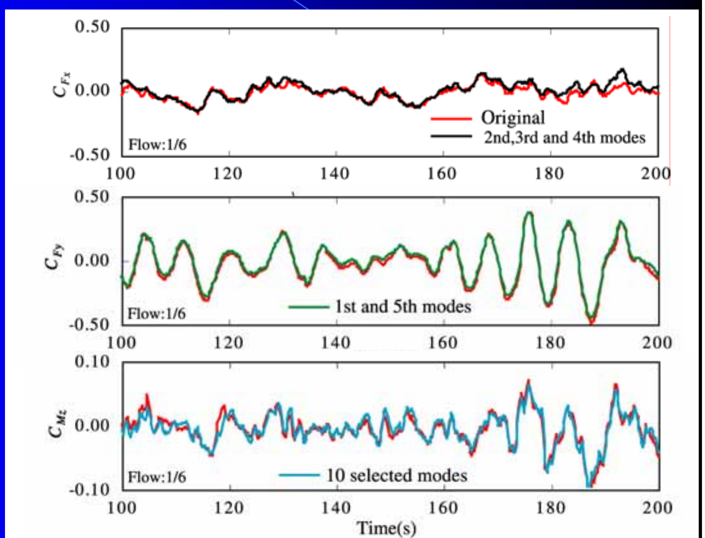


Generalized Wind Forces Reconstructed from Selected Dominant Modes

Along-wind Force

Across-wind Force

Torsional Moment



Response Analysis in Time Domain

Analytical Condition

- Coupled oscillation of along-wind, across-wind and torsional components
- Newmark β method : $\beta = 1/4$
- Time interval : $\Delta t = 0.271$s
- Calculation length : $T = 600$s
- Tip mean wind speed : (H=200m) $V_H = 55$m/s (100y-recurrence in Tokyo)

Analytical Model and Forces

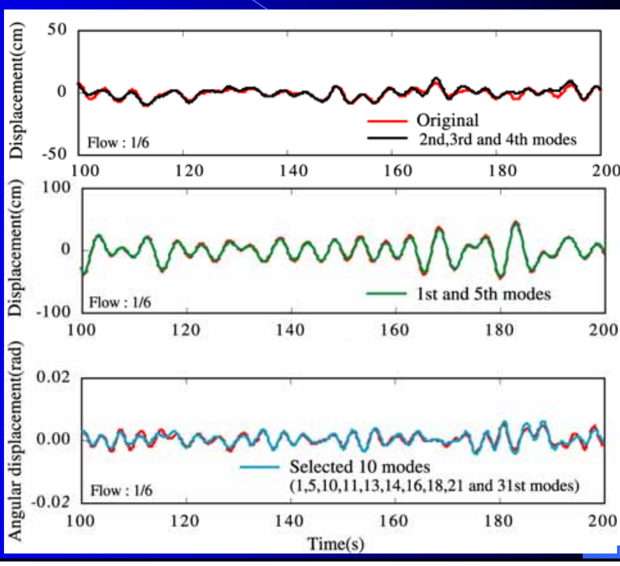
- 25 Lumped masses ×3DOF
- Fundamental natural periods : $T_{1X} = T_{1Y} = 5$s, $T_{1\theta} = 1.3$s
- Damping ratios : 2% to the critical
- Wind forces:
  - based on original pressures
  - based on reconstructed pressures by selected dominant modes
    - Along-wind : 2nd, 3rd & 4th
    - Across-wind : 1st & 5th
    - Torsional : 1st, 5th ....31st (10 modes)

Responses due to Wind Forces Reconstructed from Selected Dominant Modes

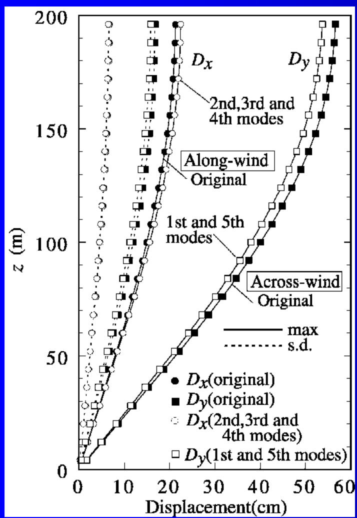
Along-wind Displacement

Across-wind Displacement

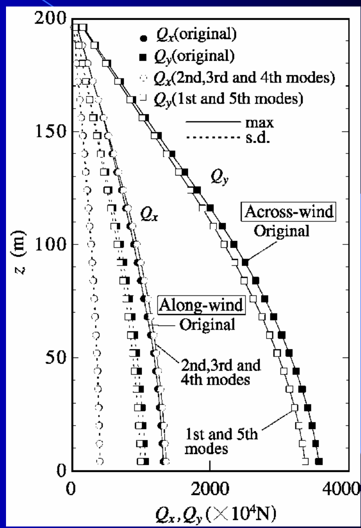
Torsional Angle



Responses due to Wind Forces Reconstructed from Selected Dominant Modes



Displacement



Shear Force

Responses due to Wind Forces Reconstructed from Selected Dominant Modes

Wind Forces	Response at Top (H=200m)					
	Along-wind Displacement (cm)		Across-wind Displacement (cm)		Angular Displacement (cm)	
	Max	S.D.	Max	S.D.	Max	S.D.
Original	21.3	6.67	56.6	16.8	0.0107	0.0025
Reconstructed from Selected Dominant Modes	22.5	6.46	53.7	15.9	0.0112	0.0026
	(2nd, 3rd and 4th)		(1st and 5th)		(10 selected*)	
Error (%)	5.6	3.1	5.1	5.7	4.8	3.4

* 1st, 5th, 10th, 11th, 13th, 14th, 16th, 18th, 21st and 31st Modes

Coordinate Transformation Matrix [*A*]

Matrix [*A*] as an operator for transforming the coordinate

$$\{b\} = [A]\,\{a\}$$

- **Vector $\{a\}$ is transformed to another Vector $\{b\}$ of a different magnitude and a different direction by operation of Matrix [*A*].**

Eigenvalue and Eigenvector

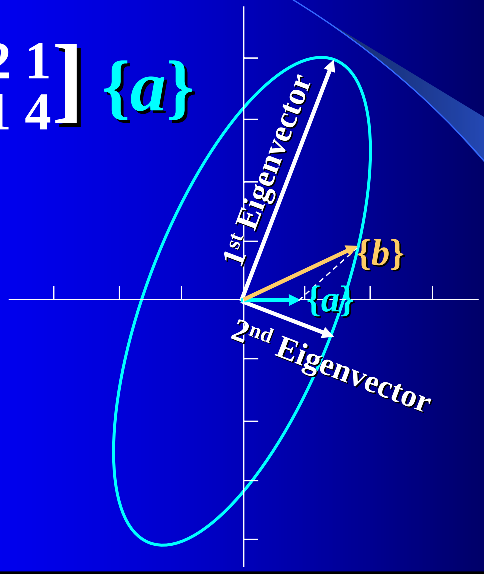
ex. $\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$

Eigenvalues : λ = 4.41, 1.59

Eigenvectors: $\{1, 2.41\}^T$, $\{1, -0.41\}^T$

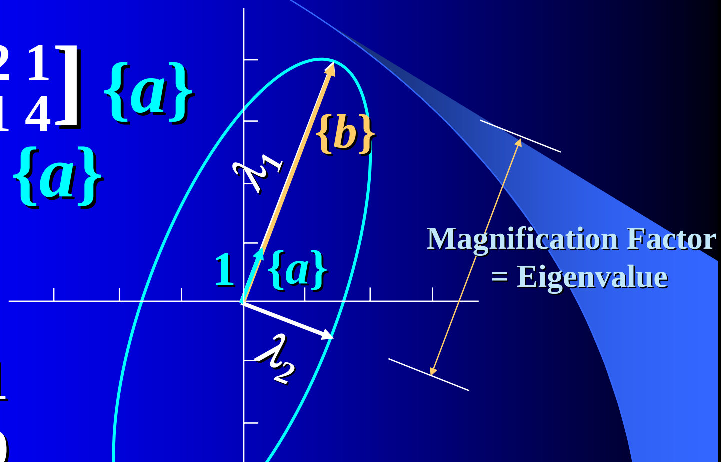
Coordinate Transformation Matrix and Eigenvalue / Eigenvector

$$\{b\} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \{a\}$$



Coordinate Transformation Matrix and Eigenvalue / Eigenvector

$$\{b\} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \{a\}$$
$$= \lambda \{a\}$$



$\lambda_1 = 4.41$

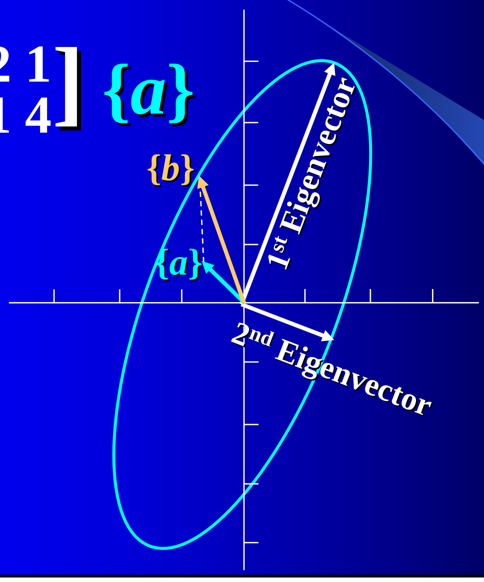
$\lambda_2 = 1.59$

a : Unit Vector

$|a| = 1$

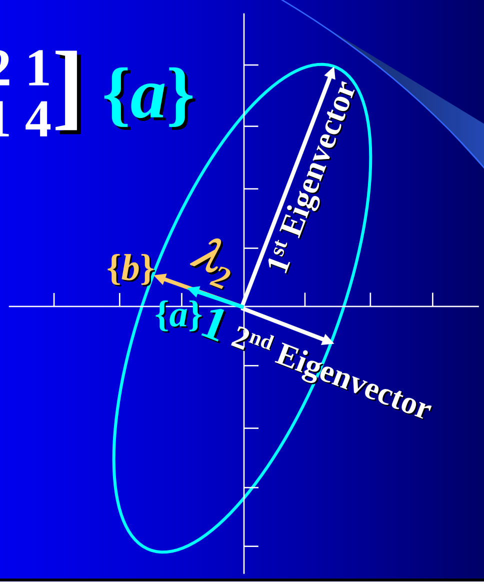
Coordinate Transformation Matrix and Eigenvalue / Eigenvector

$$\{b\} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \{a\}$$



Coordinate Transformation Matrix and Eigenvalue / Eigenvector

$$\{b\} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \{a\}$$



POD of Random Field with a Singular Condition

$$ex. \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = [R_p]$$

Eigenvalues : $\lambda = a$ (multiple root)

Eigenvectors: *Indeterminate !*

Uncorrelated with the same variances!

Example of Eigenvectors

Sample A

1st Mode

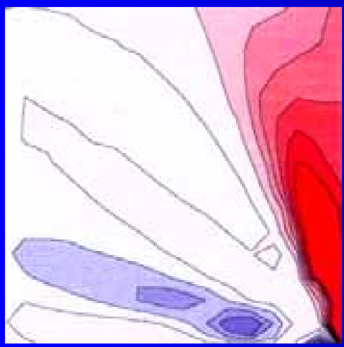
$c_1 = 29.2\%$

2nd Mode

$c_2 = 25.4\%$

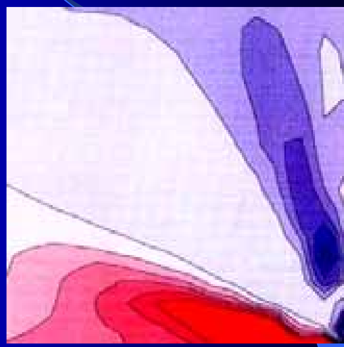
Example of Eigenvectors

Sample B



1st Mode

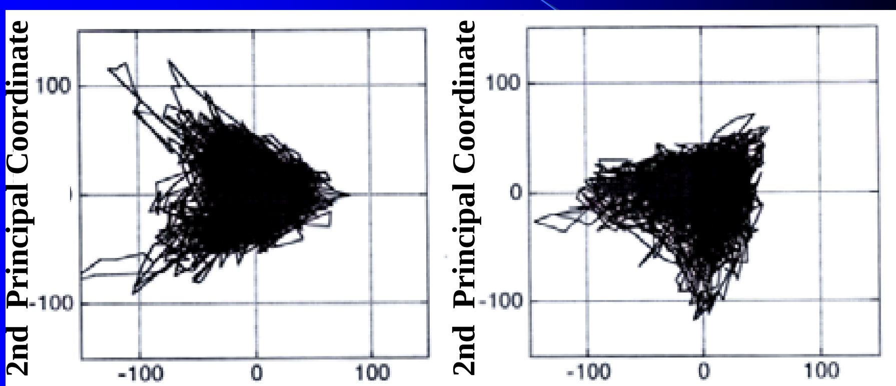
$c_1 = 30.6\%$



2nd Mode

$c_2 = 24.7\%$

State Locus by $a_1(t)$ and $a_2(t)$



1st Principal Coordinate

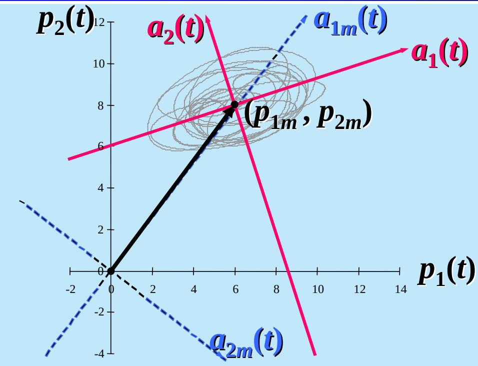
Sample A

1st Principal Coordinate

Sample B

Eigenvalues and Proportions with/without Inclusion of Mean Value

	Without Mean Value		With Mean Value	
Mode	Eigenvalue	Proportion (%)	Eigenvalue	Proportion (%)
1st	4.11	80.1	103.11	98.1
2nd	1.02	19.9	2.03	1.9



Merits of POD

- Observe phenomena by most efficient coordinates
- Extract hidden systematic structures from random information
- A significant reduction in amount of information that needs to be stored